TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 289, Number 2, June 1985

THE TYPICAL STRUCTURE OF THE SETS $\{x: f(x) = h(x)\}$ FOR f CONTINUOUS AND h LIPSCHITZ

BY

ZYGMUNT WÓJTOWICZ

ABSTRACT. Let R be the space of real numbers and C the space of continuous functions f: $[0,1] \to R$ with the uniform norm. Bruckner and Garg prove that there exists a residual set B in C such that for every function $f \in B$ there exists a countable dense set Λ_f in R such that: for $\lambda \notin \Lambda_f$ the top and bottom levels in the direction λ of f are singletons, in between these levels there are countably many levels in the direction λ of f that consist of a nonempty perfect set together with a single isolated point, and the remaining levels in the direction λ of f are all perfect; for $\lambda \in \Lambda_f$ the level structure in the direction λ of f is the same except that one (and only one) of the levels has two isolated points instead of one.

In this paper we show that the analogue of the above theorem holds: if we replace the family of straight lines $\{\lambda x + c\}$ by a 2-parameter family H that is almost uniformly Lipschitz; and if we replace $\{\lambda x + c\}$ by a homeomorphical image of a certain 2-parameter family H that is almost uniformly Lipschitz.

1. Introduction. Let R be the space of real numbers and C the space of continuous functions f: $[0,1] \to R$ equipped with the uniform norm

$$\|f\| = \sup\{|f(x)|: 0 \leqslant x \leqslant 1\}.$$

A subset A of C is *residual* in C if its complement $C \setminus A$ is of the first category in C. If $f \in C$ and $\varepsilon > 0$, the open ball $\{g \in C: \|g - f\| < \varepsilon\}$ of C is denoted, as usual, by $B(f, \varepsilon)$.

An interval $I \subset [0,1]$ is *rational* if both its endpoints are rational, and *open* if it is open relative to $[0,1]$.

Let f be a given function in C and let $\lambda \in R$. For every $c \in R$ the set $\{x: f(x) = \lambda x + c\}$ is called a level of f in the direction λ. By a level of f we mean, in general, a level of f in some direction $\lambda \in R$.

Let

$$a_{f,\lambda} = \inf\{f(x) - \lambda x: 0 \leqslant x \leqslant 1\}$$

and

$$b_{f,\lambda} = \sup\{f(x) - \lambda x: 0 \leqslant x \leqslant 1\}.$$

1. DEFINITION. The levels of a function f in C are *normal in a direction* $\lambda \in R$ if there exists a countable dense set $E_{f,\lambda}$ in $(a_{f,\lambda}, b_{f,\lambda})$ such that the level $\{x: f(x) = \lambda x + c\}$ of f in the direction λ is

(a) a perfect set when $c \notin E_{f,\lambda} \cup \{a_{f,\lambda}, b_{f,\lambda}\}$,

Received by the editors May 7, 1982.

1980 *Mathematics Subject Classification*. Primary 26A15; Secondary 26A27.

(b) a single point when $c = a_{f,\lambda}$ or $c = b_{f,\lambda}$, and

(c) the union of a nonempty perfect set P with an isolated point $x \notin P$ when $c \in E_{f,\lambda}$ (P and x depend on f, λ and c).

A. M. Bruckner and K. M. Garg have proved [**1**, Theorem 4.8] that there exists a residual set of functions f in C such that the levels of f are normal in all but a countable dense set of directions Λ_f in R, and in each direction $\lambda \in \Lambda_f$ the levels of f are normal except that there is a unique element $c \in E_{f,\lambda} \cup \{a_{f,\lambda}, b_{f,\lambda}\}$ for which the level $\{x: f(x) = \lambda x + c\}$ contains two isolated points instead of one.

2. DEFINITION. A family of functions $H \subset C$ is a 2-*parameter family* if for every pair of numbers $x_1, x_2 \in [0,1]$ $(x_1 \neq x_2)$ and for every pair of numbers $y_1, y_2 \in R$ there exists a unique function h in H such that $h(x_1) = y_1$ and $h(x_2) = y_2$.

In [**2**] Bruckner and Garg raised the following question. What conditions on H will guarantee that the analogue of the above theorem holds on replacing the family of straight lines $\{\lambda x + c\}$ by H?

In this paper we show that this question has an affirmative answer in two cases:

(1) if H is almost uniformly Lipschitz; or

(2) if H is a homeomorphical image of a certain 2-parameter family H' that is almost uniformly Lipschitz.

For the proof of this fact we use the methods of Bruckner and Garg [**1**].

2. Properties of a 2-parameter family. Let H denote a 2-parameter family of continuous functions. A function h in H for which $c = h(0)$ and $\lambda = h(1) - h(0)$ will be denoted by $h_{\lambda,c}$. The number λ is called the increase of the function $h_{\lambda,c}$.

If $H_\lambda = \{h \in H: h(1) - h(0) = \lambda\}$, then clearly $H_{\lambda_1} \cap H_{\lambda_2} = \varnothing$ for $\lambda_1 \neq \lambda_2$ and $\bigcup_{\lambda \in R} H_\lambda = H$.

1. PROPOSITION. *If* $x_0 \in [0,1]$, $y_0 \in R$ *and* h, $h_1 \in H$, $h \neq h_1$, *are such that* $h(x_0) = h_1(x_0) = y_0$, *then either* $h(x) < h_1(x)$ *when* $0 \leqslant x < x_0$ *and* $h(x) > h_1(x)$ *when* $x_0 < x \leqslant 1$; *or* $h(x) > h_1(x)$ *when* $0 \leqslant x < x_0$ *and* $h(x) < h_1(x)$ *when* $x_0 < x \leqslant 1$.

PROOF. It is clear for $x_0 = 0$ and $x_0 = 1$. Let $x_0 \in (0,1)$. Since h, $h_1 \in H$, they satisfy exactly one of the following conditions:

1°. $h(x) < h_1(x)$ when $0 \leqslant x < x_0$ and $h(x) > h_1(x)$ when $x_0 < x \leqslant 1$;

2°. $h(x) > h_1(x)$ when $0 \leqslant x < x_0$ and $h(x) < h_1(x)$ when $x_0 < x \leqslant 1$;

3°. $h(x) < h_1(x)$ when $0 \leqslant x < x_0$ and $h(x) < h_1(x)$ when $x_0 < x \leqslant 1$;

4°. $h(x) > h_1(x)$ when $0 \leqslant x < x_0$ and $h(x) > h_1(x)$ when $x_0 < x \leqslant 1$.

We show that 3° and 4° are impossible. Let h and h_1 satisfy 3° and let $h_2 \in H$, $h_2(0) = \frac{1}{2}(h(0) + h_1(0))$ and $h_2(1) = h_1(1)$. Clearly $h_2(x_0) \neq h_1(x_0)$. In the interval $(0, x_0)$ there exists a point t_1 such that $h_2(t_1) = h(t_1)$ or $h_2(t_1) = h_1(t_1)$. If $h_2(t_1) = h_1(t_1)$, then $h_1 = h_2$, which is impossible. If $h_2(t_1) = h(t_1)$, there exists a point $t_2 \in (x_0, 1)$ such that $h_2(t_2) = h(t_2)$; hence $h_2 = h$, which is impossible.

Similarly, h and h_1 do not satisfy 4°. This completes the proof.

2. PROPOSITION. *For every triple of numbers* $x_0 \in [0,1]$ *and* y_0, $\lambda \in R$, *there exists a unique function* $h \in H_\lambda$ *such that* $h(x_0) = y_0$.

PROOF. Let y_0, $\lambda \in R$. It is clear for $x_0 = 0$ and $x_0 = 1$. Let $x_0 \in (0,1)$ and let $c_0 = \sup\{c \in R: h_{\lambda,c}(x_0) < y_0\}$. It is easy to see that $h_{\lambda,c_0}(x_0) = y_0$. Suppose there exist functions h and h_1 in H_λ such that $h \neq h_1$ and $h(x_0) = y_0 = h_1(x_0)$. We have $\lambda = h(1) - h(0) = h_1(1) - h_1(0)$. According to Proposition 1, either $h_1(0) < h(0)$ and $h_1(1) > h(1)$, or $h_1(0) > h(0)$ and $h_1(1) < h(1)$. Hence, either $h_1(1) - h_1(0) > h(1) - h(0)$, or $h_1(1) - h_1(0) < h(1) - h(0)$, which is impossible. This proves $h = h_1$.

From Proposition 2 we obtain

1. COROLLARY. *If $h_1, h_2 \in H_\lambda$ and $h_1 \neq h_2$, then $h_1(x) \neq h_2(x)$ for every $x \in [0,1]$. In particular, $h_1(0) > h_2(0)$ if and only if $h_1(x) > h_2(x)$ for every $x \in [0,1]$.*

1. LEMMA. *If $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} \lambda_n = \lambda$, then $\lim_{n\to\infty} h_{\lambda_n,c_n}(x) = h_{\lambda,c}(x)$ for every $x \in [0,1]$.*

PROOF. Clearly $\lim_{n\to\infty} h_{\lambda_n,c_n}(0) = h_{\lambda,c}(0)$ and $\lim_{n\to\infty} h_{\lambda_n,c_n}(1) = h_{\lambda,c}(1)$. Suppose there exists a number $x_0 \in (0,1)$ such that $h_{\lambda,c}(x_0)$ is not a limit of the sequence $\{h_{\lambda_n,c_n}(x_0)\}$. Then there exists an $\varepsilon > 0$ such that the set

$$\left\{h_{\lambda_n,c_n}(x_0)\right\} \setminus \left(h_{\lambda,c}(x_0) - \varepsilon, h_{\lambda,c}(x_0) + \varepsilon\right)$$

is infinite. According to Proposition 2 there exist functions $h_{\lambda,c'}, h_{\lambda,c''} \in H$ such that $h_{\lambda,c'}(x_0) = h_{\lambda,c}(x_0) - \varepsilon$ and $h_{\lambda,c''}(x_0) = h_{\lambda,c}(x_0) + \varepsilon$. Hence, and from Corollary 1, $c' < c''$. There exists a natural number N such that $c_n \in (c', c'')$ for every $n > N$. For a function h_{λ_n,c_n} such that $n > N$ and $h_{\lambda_n,c_n}(x_0) \notin (h_{\lambda,c}(x_0) - \varepsilon, h_{\lambda,c}(x_0) + \varepsilon)$, we have

$$h_{\lambda_n,c_n}(1) \notin \left(h_{\lambda,c'}(1), h_{\lambda,c''}(1)\right).$$

The above yields that $\{h_{\lambda_n,c_n}(1)\} \setminus (h_{\lambda,c'}(1), h_{\lambda,c''}(1))$ is infinite, which contradicts the assumption that $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty} \lambda_n = \lambda$, so $\lim_{n\to\infty} h_{\lambda_n,c_n}(x) = h_{\lambda,c}(x)$ for every $x \in [0,1]$.

From Dini's Theorem and Lemma 1 we obtain

2. LEMMA. *Let $\lambda \in R$. Then $\lim_{n\to\infty} \|h_{\lambda,c_n} - h_{\lambda,c}\| = 0$ if and only if $\lim_{n\to\infty} c_n = c$.*

3. PROPOSITION. $\lim_{n\to\infty} \|h_{\lambda_n,c_n} - h_{\lambda,c}\| = 0$ *if and only if* $\lim_{n\to\infty} c_n = c$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$.

PROOF. Necessity is trivial. For the sufficiency assume $\lim_{n\to\infty} \lambda_n = \lambda$, $\lim_{n\to\infty} c_n = c$ and zero is not the limit of the sequence $\{\|h_{\lambda_n,c_n} - h_{\lambda,c}\|\}$. Then there is an $\varepsilon > 0$ such that for every natural number N there exist numbers $n_N > N$ and $x_N \in (0,1)$ such that

$$\left|h_{\lambda_{n_N},c_{n_N}}(x_N) - h_{\lambda,c}(x_N)\right| \geqslant \varepsilon.$$

According to Lemma 2 for ε there exist $h_{\lambda,c'}, h_{\lambda,c''} \in B(h_{\lambda,c}, \varepsilon)$ such that $h_{\lambda,c'}(x) < h_{\lambda,c}(x)$ and $h_{\lambda,c''}(x) > h_{\lambda,c}(x)$ for every $x \in [0,1]$. Let N be a natural number such that $h_{\lambda_n,c_n}(0) \in (c', c'')$ and $h_{\lambda_n,c_n}(1) \in (h_{\lambda,c'}(1), h_{\lambda,c''}(1))$ for every $n > N$. For $h_{\lambda_{n_N},c_{n_N}}$ we have

$$h_{\lambda_{n_N},c_{n_N}}(x_N) \geqslant h_{\lambda,c}(x_N) + \varepsilon > h_{\lambda,c}(x_N)$$

or

$$h_{\lambda_{n_N},c_{n_N}}(x_N) \leqslant h_{\lambda,c}(x_N) - \varepsilon < h_{\lambda,c}(x_N).$$

Hence,

$$h_{\lambda_{n_N}, c_{n_N}} = h_{\lambda, c'} \quad \text{or} \quad h_{\lambda_{n_N}, c_{n_N}} = h_{\lambda, c''},$$

which contradicts the assumption that $h_{\lambda_{n_N}, c_{n_N}}(0) \in (c', c'')$; therefore

$$\lim_{n\to\infty} \|h_{\lambda_n, c_n} - h_{\lambda, c}\| = 0.$$

4. PROPOSITION. *For every natural number* n, *let* (x'_n, y'_n), (x''_n, y''_n), (x', y'), $(x'', y'') \in [0,1] \times R$ $(x'_n \neq x''_n$ *and* $x' \neq x'')$ *and let* $h_{\lambda_n, c_n}, h_{\lambda, c} \in H$ *be functions such that* $h_{\lambda_n, c_n}(x'_n) = y'_n$, $h_{\lambda_n, c_n}(x''_n) = y''_n$, $h_{\lambda, c}(x') = y'$ *and* $h_{\lambda, c}(x'') = y''$.

Then if $\lim_{n\to\infty}(x'_n, y'_n) = (x', y')$ *and* $\lim_{n\to\infty}(x''_n, y''_n) = (x'', y'')$, *we have* $\lim_{n\to\infty}\|h_{\lambda_n, c_n} - h_{\lambda, c}\| = 0$.

PROOF. Assume $x' < x''$. Let $\varepsilon > 0$ and let x_0 be a number such that $x' < x_0 < x''$. According to Propositions 1 and 3 there exist functions $h_{\lambda', c'}, h_{\lambda'', c''} \in H$ such that

$$h_{\lambda', c'}(x_0) = h_{\lambda, c}(x_0) = h_{\lambda'', c''}(x_0), \qquad c' \in (c, c+\varepsilon), c'' \in (c-\varepsilon, c)$$

and

$$h_{\lambda', c'}(1), h_{\lambda'', c''}(1) \in \big(h_{\lambda, c}(1) - \varepsilon, h_{\lambda, c}(1) + \varepsilon\big).$$

Let d_1 (d_2) be the distance from the point (x', y') (x'', y'') to the graphs of $h_{\lambda', c'}$ and $h_{\lambda'', c''}$. There is a natural number N such that $(x'_n, y'_n) \in K((x', y'), d_1) \cap ([0,1] \times R)$ and $(x''_n, y''_n) \in K((x'', y''), d_2) \cap ([0,1] \times R)$ for every $n > N$, where K is the open ball in R^2. By Proposition 1, for $n > N$ there exist numbers $t'_n \in (0, x_0)$ and $t''_n \in (x_0, 1)$ such that

$$h_{\lambda'', c''}(x) < h_{\lambda_n, c_n}(x) < h_{\lambda', c'}(x) \quad \text{for } x < t'_n$$

and

$$h_{\lambda'', c''}(x) > h_{\lambda_n, c_n}(x) > h_{\lambda', c'}(x) \quad \text{for } x > t''_n.$$

In particular,

$$c'' < c_n < c' \quad \text{and} \quad h_{\lambda', c'}(1) < h_{\lambda_n, c_n}(1) < h_{\lambda'', c''}(1).$$

Hence, $\lim_{n\to\infty} c_n = c$ and $\lim_{n\to\infty}\lambda_n = \lambda$, so, by Proposition 3,

$$\lim_{n\to\infty} \|h_{\lambda_n, c_n} - h_{\lambda, c}\| = 0.$$

3. The structure of the sets $\{x: f(x) = h(x)\}$ for f continuous and h Lipschitz.

3. DEFINITION. Let $f \in C$, $h \in H$ and let I be a subinterval of $[0,1]$. The graph of h *supports* the graph of f in I from above (below) if $h(x) \geqslant f(x)$ $(h(x) \leqslant f(x))$ for every $x \in I$ and there is a point x_0 in I such that $h(x_0) = f(x_0)$. Further, if x_0 is not unique, then the graph of h supports the graph of f in I from above (below) at more than one point. We say that the graph of h supports the graph of f in I if the graph of h supports the graph of f from above or below.

If I and J are two disjoint subintervals of $[0,1]$, the graph of h supports the graph of f in I and J if it supports the graph of f in I as well as the graph of f in J. Similarly for three or more mutually disjoint subintervals of $[0,1]$.

Following an argument parallel to that of [**1**, Lemma 4.2], we obtain:

3. LEMMA. *For every function f in C there is at most a countable set of functions in H whose graphs support the graph of f in two (or more) disjoint open subintervals of* $[0, 1]$.

Let $f \in C$ and $\lambda \in R$. We set

$$\alpha_{f,\lambda} = \inf\{c \in R: \{x: f(x) = h_{\lambda,c}(x)\} \neq \varnothing\},$$
$$\beta_{f,\lambda} = \sup\{c \in R: \{x: f(x) = h_{\lambda,c}(x)\} \neq \varnothing\}.$$

4. LEMMA. *For every function f in C and every number λ in R, graphs of the functions $h_{\lambda,\alpha_{f,\lambda}}$ and $h_{\lambda,\beta_{f,\lambda}}$ support the graph of f in* $[0,1]$ *from below and from above, respectively, at least at one point.*

PROOF. Let $f \in C$ and $\lambda \in R$. Since $h_{\lambda,\alpha_{f,\lambda}}(x) \leqslant f(x)$ and $h_{\lambda,\beta_{f,\lambda}}(x) \geqslant f(x)$ for every $x \in [0,1]$, it suffices to show that $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\} \neq \varnothing$ and $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\} \neq \varnothing$. Assume $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\} = \varnothing$. Then for every $x \in [0,1]$ we have $h_{\lambda,\alpha_{f,\lambda}}(x) - f(x) < 0$. Let

$$d = \min\{f(x) - h_{\lambda,\alpha_{f,\lambda}}(x): 0 \leqslant x \leqslant 1\}.$$

By Proposition 3 there is a function $h_{\lambda,c_1} \in B(h_{\lambda,\alpha_{f,\lambda}}, d)$ such that $h_{\lambda,c_1} \neq h_{\lambda,\alpha_{f,\lambda}}$ and $c_1 > \alpha_{f,\lambda}$. Clearly, $h_{\lambda,c_1}(x) < f(x)$ for every $x \in [0,1]$, and $\{x: f(x) = h_{\lambda,c}(x)\} = \varnothing$ for every $c \in (\alpha_{f,\lambda}, c_1)$. Since this contradicts the definition of $\alpha_{f,\lambda}$ we have $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\} \neq \varnothing$. Similarly we show that $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\} \neq \varnothing$.

5. LEMMA. *For every function f in C there is at most a countable set $\Lambda_f \subset R$ such that for every $\lambda \in R \setminus \Lambda_f$:*

(a) *the sets $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\}$ and $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\}$ consist of single points, and*

(b) *the set $E_{f,\lambda}$ of numbers c, such that the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is not perfect, is dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$.*

PROOF. Let $f \in C$. We denote by Λ_f the set of increases of functions in H whose graphs support the graph of f in at least two disjoint open subintervals of $[0,1]$. Then Λ_f is at most countable, by Lemma 3. Let $\lambda \in R \setminus \Lambda_f$. As the graphs of $h_{\lambda,\alpha_{f,\lambda}}$ and $h_{\lambda,\beta_{f,\lambda}}$ support the graph of f at a unique point, part (a) is now obvious.

Let (a, b) be any open subinterval of $(\alpha_{f,\lambda}, \beta_{f,\lambda})$ and let I be an open component of the open set $G = \{x: h_{\lambda,a}(x) < f(x) < h_{\lambda,b}(x)\}$. For every $c \in (a, b)$ the function $f - h_{\lambda,c} \neq 0$ in every subinterval J of I. Besides:

1°. for every $c \in (a, b)$ the set $\{x: f(x) = h_{\lambda,c}(x)\} \cap I$ consists of a single point, or

2°. there exists a number $c \in (a, b)$ such that the set $\{x: f(x) = h_{\lambda,c}(x)\} \cap I$ contains at least two different points.

In 1°, for every $c \in (a, b)$, $x_0 \in \{x: f(x) = h_{\lambda,c}(x)\} \cap I$ is the isolated point of $\{x: f(x) = h_{\lambda,c}(x)\}$ and $(a, b) \subset E_{f,\lambda}$.

In 2° there exist numbers $c \in (a, b)$ and $x_1, x_2 \in I$ $(x_1 < x_2)$ such that $f(x_1) = h_{\lambda,c}(x_1)$ and $f(x_2) = h_{\lambda,c}(x_2)$. Let

$$c_1 = \inf\{c \in R: \{x \in [x_1, x_2]: f(x) = h_{\lambda,c}(x)\} \neq \varnothing\}$$

and

$$c_2 = \sup\{c \in R: \{x \in [x_1, x_2]: f(x) = h_{\lambda,c}(x)\} \neq \varnothing\}.$$

Then as in the proof of Lemma 4 we show there are numbers t', $t'' \in [x_1, x_2]$ such that $f(t') = h_{\lambda,c_1}(t')$ and $f(t'') = h_{\lambda,c_2}(t'')$. Hence, $c_1, c_2 \in (a, b)$. Because $\lambda \notin \Lambda_f$, there exists a number $x' \in (x_1, x_2)$ such that $f(x') = h_{\lambda,c_1}(x')$ and $f(x) > h_{\lambda,c_1}(x)$ for every $x \in (x_1, x_2)\setminus\{x'\}$, or there exists a number $x'' \in (x_1, x_2)$ such that $f(x'') = h_{\lambda,c_2}(x'')$ and $f(x) < h_{\lambda,c_2}(x)$ for every $x \in (x_1, x_2)\setminus\{x''\}$. Hence the graph of h_{λ,c_1} or h_{λ,c_2} supports the graph of f in (x_1, x_2). Thus x' is the isolated point of $\{x: f(x) = h_{\lambda,c_1}(x)\}$ or x'' is the isolated point of $\{x: f(x) = h_{\lambda,c_2}(x)\}$. Hence, $c_1 \in E_{f,\lambda}$ or $c_2 \in E_{f,\lambda}$ and $(a, b) \cap E_{f,\lambda} \neq \varnothing$. This proves that $E_{f,\lambda}$ is dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$.

6. LEMMA. *There exists a residual set of functions f in C such that for every open rational interval $I \subset [0,1]$ the increases of functions in H, of which the graphs support the graph of f in I from above at more than one point, form a dense set in R, and the increases of functions in H, of which the graphs support the graph of f in I from below at more than one point, form a dense set in R.*

PROOF. It suffices to prove the result for a single fixed open rational interval $I \subset [0,1]$. Let $\{\lambda_n\}$ be an enumeration of the rational numbers. For natural numbers n and m, let $A_{n,m}$ denote the set of functions f in C for which there exists a function h in H with increase $\lambda \in (\lambda_n - 1/m, \lambda_n + 1/m)$ such that the graph of h supports the graph of f in I from above at more than one point. We first show that $A_{n,m}$ contains a dense open subset of C.

Let U be any nonempty open subset of C and let $f \in U$. There exists an $\varepsilon > 0$ such that $B(f, \varepsilon) \subset U$. Let

$$\alpha_n = \sup\Big\{c \in R: \Big\{x \in I: f(x) = h_{\lambda_n,c}(x)\Big\} \neq \varnothing\Big\}.$$

According to Lemma 2 there exists a function h_{λ_n,c_1} in H such that $h_{\lambda_n,c_1} \in B(h_{\lambda_n,\alpha_n}, \varepsilon/4)$ and $h_{\lambda_n,c_1}(x) < h_{\lambda_n,\alpha_n}(x)$ for every $x \in [0,1]$. Let J be a subinterval of I such that $f > h_{\lambda_n,c_1}$ on J. Then we can find five points x_i $(i = 0,1,\dots,4)$ in J such that $x_0 < x_1 < \cdots < x_4$. By Lemma 2 there is a function $h_{\lambda_n,c}$ in H such that $h_{\lambda_n,c} \in B(h_{\lambda_n,\alpha_n}, \varepsilon/2)$ and $h_{\lambda_n,c}(x) > h_{\lambda_n,\alpha_n}(x)$ for every x $\in [0,1]$. We set

$$d = \min\Big\{h_{\lambda_n,c}(x) - h_{\lambda_n,\alpha_n}(x): 0 \leqslant x \leqslant 1\Big\}.$$

Let h_{λ_n,c_2} in H be a function such that $h_{\lambda_n,c_2} \in B(h_{\lambda_n,\alpha_n}, d/2)$ and $h_{\lambda_n,c_2}(x) > h_{\lambda_n,\alpha_n}(x)$ for every $x \in [0,1]$, and let

$$d_1 = \min\Big\{h_{\lambda_n,c_2}(x) - h_{\lambda_n,\alpha_n}(x): 0 \leqslant x \leqslant 1\Big\}.$$

We define a function g in C by $g(x) = f(x)$ for $x \in [0, x_0] \cup [x_4, 1]$, $g(x_1) = h_{\lambda_n,c}(x_1)$, $g(x_3) = h_{\lambda_n,c}(x_3)$, $g(x_2) = h_{\lambda_n,\alpha_n}(x_2)$ and $g|_{[x_i,x_{i+1}]} = h|_{[x_i,x_{i+1}]}$ $(i = 0,1,\dots,3)$ for any h in H. Then $g \in B(f, \varepsilon)$. According to Proposition 3 and 4 there exist functions $h_{\lambda',c'}$ and $h_{\lambda'',c''}$ in H such that $h_{\lambda',c'}(x_2) = h_{\lambda'',c''}(x_2) = h_{\lambda_n,c}(x_2)$, $c_2 < c' < c$, $\lambda_n < \lambda' < \lambda_n + 1/m$, $h_{\lambda_n,c_2}(1) < h_{\lambda'',c''}(1) < h_{\lambda_n,c}(1)$ and $\lambda_n - 1/m < \lambda'' < \lambda_n$. We denote by d_2 the distance from the point $(x_1, h_{\lambda_n,c}(x_1))$ to the graph of

$h_{\lambda',c'}$, and by d_3 the distance from the point $(x_3, h_{\lambda_n,c}(x_3))$ to the graph of $h_{\lambda'',c''}$. Let

$$d_4 = \min\{h_{\lambda'',c''}(x) - g(x): x_0 \leqslant x \leqslant x_2\},$$
$$d_5 = \min\{h_{\lambda',c'}(x) - g(x): x_2 \leqslant x \leqslant x_4\}$$

and

$$\eta = \min\{d_1, d_2, d_3, d_4, d_5, \varepsilon/8\}.$$

Clearly, $B(g, \eta) \subset B(f, \varepsilon)$.

Now we show that $B(g, \eta) \subset A_{n,m}$. Let $s \in B(g, \eta)$ and $W = \{(x, y): x_0 \leqslant x \leqslant x_4, y = s(x)\}$. The points $Q, S \in W$ $(Q \neq S)$ belong to the graph of a unique function in H. Let us take the part of this graph contained between Q and S. Let T be the union of these parts for all pairs (Q, S). We show that T is closed. Let $\{(x_n, y_n)\} \subset T$ be any sequence of points converging to some point (x_0, y_0). For (x_n, y_n) there exist two different points $Q_n, S_n \in W$ such that (x_n, y_n) belongs to the curve of endpoints Q_n and S_n. There is an increasing sequence $\{n_k\}$ of natural numbers such that $\lim_{k\to\infty} Q_{n_k} = Q'$ and $\lim_{k\to\infty} S_{n_k} = S'$, where $Q', S' \in W$.

Suppose first that $Q' \neq S'$. Let $h_{q_{n_k}, s_{n_k}}$ denote the function in H for which the graph contains Q_{n_k} and S_{n_k}, and let $h_{q,s}$ denote the function in H for which the graph contains Q' and S'. We assume $(x_0, y_0) \notin T$. Let d' be the distance from (x_0, y_0) to the graph of $h_{q,s}$. By Proposition 4, $\lim_{k\to\infty} \|h_{q_{n_k}, s_{n_k}} - h_{q,s}\| = 0$. Hence there exists a natural number N such that

$$(x_{n_k}, y_{n_k}) \in \left\{(x, y): h_{q,s}(x) - \frac{d'}{3} < y < h_{q,s}(x) + \frac{d'}{3}, 0 \leqslant x \leqslant 1\right\}$$

and

$$(x_{n_k}, y_{n_k}) \in K((x_0, y_0), d'/3) \quad \text{for } k > N.$$

Since this contradicts the hypothesis that $\lim_{n\to\infty}(x_n, y_n) = (x_0, y_0)$, we have $(x_0, y_0) \in T$.

Now suppose $Q' = S'$ and $(x_0, y_0) \notin T$. Let a point (x', y') belong to the interior of the segment of endpoints Q' and (x_0, y_0), and let h_1 be a function in H such that the $h_1(x') = y'$ and the graph of h_1 lies between Q' and (x_0, y_0). We let r_1 be the distance from Q' to the graph of h_1, and r_2 the distance from (x_0, y_0) to the graph of h_1. Then there exists a natural number N such that $Q_{n_k}, S_{n_k} \in K(Q', r_1)$ for every $k > N$. Hence, $(x_{n_k}, y_{n_k}) \notin K((x_0, y_0), r_2)$ for $k > N$. Since this contradicts the hypothesis that $\lim_{n\to\infty}(x_n, y_n) = (x_0, y_0)$, we have $(x_0, y_0) \in T$. This proves that T is closed and compact.

Let P be the maximum point of T belonging to the line $x = x_2$. There exist points Q_0 and S_0 on the graph of s in $[x_0, x_2)$ and on the graph of s in $(x_2, x_4]$, respectively, such that P belongs to the graph of the function h_{q_0,s_0} in H. The graph of h_{q_0,s_0} supports the graph of s in $[x_0, x_4]$ from above. We have

$$h_{\lambda',c'}(1) > h_{\lambda,c_3}(1) > h_{\lambda'',c''}(1), \qquad c' < c_3 < c'',$$

and

$$h_{\lambda',c'}(1) - c' > h_{\lambda,c_3}(1) - c_3 > h_{\lambda'',c''}(1) - c''.$$

Hence, $\lambda'' < \lambda < \lambda'$ and $\lambda \in (\lambda_n - 1/m, \lambda_n + 1/m)$. This proves that $s \in A_{n,m}$ and $B(g, \eta) \subset A_{n,m}$. Thus $A_{n,m}$ contains a dense open subset of C, so it is residual in C. Hence the set $A = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} A_{n,m}$ is residual in C.

For $f \in A$ and $(p, q) \subset R$ there exist a rational number $\lambda_n \in (p, q)$ and a natural number m such that $(\lambda_n - 1/m, \lambda_n + 1/m) \subset (p, q)$. Since $f \in A_{n,m}$ there exists a function $h_{\lambda,c} \in H$ for which the graph supports the graph of f in I from above at more than one point and $\lambda \in (\lambda_n - 1/m, \lambda_n + 1/m)$. This proves that the set of increases of functions in H for which the graphs support the graph of f from above at more than one point, is dense in R. Similarly we prove that the set B of functions f in C, such that the set of increases of functions in H for which the graphs support the graph of f from below at more than one point, is dense in R and residual in C. Clearly the set $D = A \cap B$ is residual in C.

Following an argument parallel to that of [**1**, Lemma 4.6 and 4.7], we obtain:

7. LEMMA. *The set of functions f in C for which the graphs support at least one function of H at more than one point is of the first category in C.*

8. LEMMA. *The set of functions f in C for which there exist $\lambda \in R$ and two different functions h_{λ,c_1} and h_{λ,c_2} in H whose graphs support the graph of f in two different points is of the first category in C.*

4. DEFINITION. A 2-parameter family H of continuous functions is *almost uniformly Lipschitz* if

$$\underset{c \in R}{\forall} \ \underset{\lambda \in R}{\forall} \ \underset{L_{\lambda,c} \geqslant 0}{\exists} \ \underset{x_1, x_2 \in [0,1]}{\forall} |h_{\lambda,c}(x_1) - h_{\lambda,c}(x_2)| \leqslant L_{\lambda,c}|x_1 - x_2|,$$

and, for every natural number n,

$$M_n = \sup\{ L_{\lambda,c} : \lambda \in [-n, n], c \in [-n, n]\} < +\infty.$$

5. DEFINITION. A function f is said to be *nondecreasing* (*nonincreasing*) at a point t if there is a positive number h such that

$$f(x) \leqslant f(t) \ (f(x) \geqslant f(t)) \quad \text{for } t - h < x < t,$$
$$f(t) \leqslant f(x) \ (f(t) \geqslant f(x)) \quad \text{for } t < x < t + h.$$

f is said to be *monotone* at t if it is nondecreasing or nonincreasing at t.

9. LEMMA. *Let H be a 2-parameter family of continuous functions which is almost uniformly Lipschitz.*

Then there exits a residual set of functions f in C such that for every function h in H the function $f - h$ is not monotone at any point $x \in [0,1]$.

PROOF. Let A denote the set of functions f in C for which there exists a function h in H such that $f - h$ is nondecreasing at some point of $[0,1]$. For every natural number n we denote by A_n the set of functions f in C for which there exist numbers $\lambda, c \in [-n, n]$ and $x \in [0,1]$ such that

1°. $f(t) - h_{\lambda,c}(t) \leqslant f(x) - h_{\lambda,c}(x)$, for $t \in [0,1] \cap (x - 1/n, x)$, and

2°. $f(x) - h_{\lambda,c}(x) \leqslant f(t) - h_{\lambda,c}(t)$, for $t \in [0,1] \cap (x, x + 1/n)$.

Clearly $A = \bigcup_{n=1}^{\infty} A_n$. First we show that A_n is closed and nowhere dense for each n.

Let n be a given natural number and let $\{f_k\}$ by any sequence of functions in A_n uniformly convergent to some function f in C. Then for each k there exist λ_k, $c_k \in [-n, n]$ and $x_k \in [0,1]$ such that

$$f_k(t) - h_{\lambda_k, c_k}(t) \leqslant f_k(x_k) - h_{\lambda_k, c_k}(x_k),$$

for $t \in [0,1] \cap (x_k - 1/n, x_k)$, and

$$f_k(x_k) - h_{\lambda_k, c_k}(x_k) \leqslant f_k(t) - h_{\lambda_k, c_k}(t),$$

for $t \in [0,1] \cap (x_k, x_k + 1/n)$. Moreover, there exists an increasing sequence $\{k_i\}$ of natural numbers such that $\{\lambda_{k_i}\}$ converges to some $\lambda \in [-n, n]$, $\{c_{k_i}\}$ converges to some $c \in [-n, n]$, and $\{x_{k_i}\}$ converges to some $x \in [0,1]$. According to Proposition 3 the sequence $\{h_{\lambda_{k_i}, c_{k_i}}\}$ converges uniformly to the function $h_{\lambda,c}$. It is easy to verify that $f(t) - h_{\lambda,c}(t) \leqslant f(x) - h_{\lambda,c}(x)$ for $t \in [0,1] \cap (x - 1/n, x)$, and $f(x) - h_{\lambda,c}(x) \leqslant f(t) - h_{\lambda,c}(t)$ for $t \in [0,1] \cap (x, x + 1/n)$. Thus $f \in A_n$ and A_n is closed in C.

To prove that A_n is nowhere dense in C, it suffices to show that A_n does not contain any open subset of C. Let U be a given nonempty open subset of C. Then there exists a polynomial $g \in U$ and an $\varepsilon > 0$ such that $B(g, \varepsilon) \subset U$. Let $0 < \alpha < \varepsilon$, $\beta = \sup\{|g(x)|: 0 \leqslant x \leqslant 1\}$, and let m be an odd integer such that $m > \max\{2M_n, 2n, 3(\beta + M_n)/\alpha\}$. Let s be a function defined by

$$s(x) = \begin{cases} \alpha & \text{for } x = \dfrac{2i}{m};\ i = 0, 1, \dots, \dfrac{m-1}{2}, \\ 0 & \text{for } x = \dfrac{2i+1}{m};\ i = 0, 1, \dots, \dfrac{m-1}{2}, \\ \text{linear function} & \text{for } x \in \left[\dfrac{i}{m}, \dfrac{i+1}{m}\right];\ i = 0, 1, \dots, m-1. \end{cases}$$

Then s is continuous and $f = g + s \in U$. We claim that $f \notin A_n$. Let λ, $c \in [-n, n]$. We have $M_n - L_{\lambda,c} \geqslant 0$. For $x \in [0,1]$ there exists an integer i such that

$$0 \leqslant i \leqslant (m-1)/2 \quad \text{and} \quad 2i/m \leqslant x \leqslant (2i+2)/m.$$

In the case $2i/m \leqslant x < (2i+1)/m$, we have

$$x < (2i+1)/m < x + 1/n$$

and

$$\begin{aligned} &\left[f\left(\frac{2i+1}{m}\right) - h_{\lambda,c}\left(\frac{2i+1}{m}\right)\right] - [f(x) - h_{\lambda,c}(x)] \\ &\quad \leqslant \beta\left(\frac{2i+1}{m} - x\right) - \alpha m\left(\frac{2i+1}{m} - x\right) + L_{\lambda,c}\left(\frac{2i+1}{m} - x\right) \\ &\quad < (-M_n + L_{\lambda,c})\left(\frac{2i+i}{m} - x\right) \leqslant 0. \end{aligned}$$

In the case $(2i+1)/m \leqslant x < (2i + \frac{3}{2})/m$, we have $x - 1/n < 2i/m < x$ and

$$\begin{aligned} &\left[f\left(\frac{2i}{m}\right) - h_{\lambda,c}\left(\frac{2i}{m}\right)\right] - [f(x) - h_{\lambda,c}(x)] \\ &\quad \geqslant \beta\left(\frac{2i}{m} - x\right) + \frac{\alpha}{2} - L_{\lambda,c}\left(x - \frac{2i}{m}\right) \\ &\quad > (-M_n + L_{\lambda,c})\left(\frac{2i}{m} - x\right) \geqslant 0. \end{aligned}$$

In the case $(2i+\frac{3}{2})/m \leqslant x < (2i+2)/m$, we have $x < (2i+3)/m < x + 1/n$ and

$$\left[f\left(\frac{2i+3}{m}\right) - h_{\lambda,c}\left(\frac{2i+3}{m}\right)\right] - [f(x) - h_{\lambda,c}(x)]$$
$$\leqslant \beta\left(\frac{2i+3}{m} - x\right) - \frac{\alpha}{2} + L_{\lambda,c}\left(\frac{2i+3}{m} - x\right)$$
$$< (-M_n + L_{\lambda,c})\left(\frac{2i+3}{m} - x\right) \leqslant 0.$$

Hence, f does not satisfy inequalities 1° and 2° at any point $x \in [0,1]$; therefore $f \notin A_n$. This proves that A_n is nowhere dense in C, and A is of the first category in C.

If B is the set of functions f in C for which there is a function $h \in H$ such that $f - h$ is nonincreasing at some point of $[0,1]$, then in the same way as for A we can prove that B is of the first category in C. From this it follows that $F = C \setminus (A \cup B)$ is a residual set.

1. THEOREM. *Let H be a 2-parameter family of continuous functions which is almost uniformly Lipschitz.*

Then there exists a residual set of functions in C for which there exist a countable dense set $\Lambda_f \subset R$ and countable dense set $E_{f,\lambda} \subset (\alpha_{f,\lambda}, \beta_{f,\lambda})$ such that:

1°. *If $\lambda \in R \setminus \Lambda_f$ then*:

(a) *the sets $\{x: f(x) = h_{\lambda,\alpha_{f,\lambda}}(x)\}$ and $\{x: f(x) = h_{\lambda,\beta_{f,\lambda}}(x)\}$ consist of single points*;

(b) *for $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda}) \setminus E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is perfect*; *and*

(c) *for $c \in E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is the union of a nonempty perfect set and an isolated point.*

2°. *If $\lambda \in \Lambda_f$ then*:

(a) *there exists a unique number $c_{f,\lambda} \in E_{f,\lambda} \cup \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ such that*: *if $c_{f,\lambda} \in E_{f,\lambda}$ then $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ is the union of a nonempty perfect set and two isolated points*; *and if $c_{f,\lambda} \in \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$, then $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ consists of two different points*;

(b) *for $c \in E_{f,\lambda} \setminus \{c_{f,\lambda}\}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is the union of a nonempty perfect set and an isolated point*;

(c) *for $c \in \{\alpha_{f,\lambda}, \beta_{f,\lambda}\} \setminus \{c_{f,\lambda}\}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ consists of a single point*; *and*

(d) *for $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda}) \setminus E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is perfect.*

PROOF. Let B, D, E and F denote sets of functions determined by Lemmas 6, 7, 8 and 9, respectively, and let

$$A = B \cap (C \setminus D) \cap (C \setminus E) \cap F.$$

Clearly A is a residual set in C. Let $f \in A$. We denote by Λ_f the set of increases of functions h in H for which the graphs support the graph of f in at least two disjoint open subintervals of $[0,1]$. According to Lemmas 4 and 6, Λ_f is a countable dense subset in R. For $\lambda \in R$ and $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$ we have

$$\{x: f(x) = h_{\lambda,c}(x)\} = \{x: f(x) - h_{\lambda,c}(x) = 0\}.$$

By Lemma 9 the function $f - h_{\lambda,c}$ is not monotone at any point of $[0,1]$. Hence, x is an isolated point of $\{x: f(x) = h_{\lambda,c}(x)\}$ if and only if a proper extremum of $f - h_{\lambda,c}$ at x is equal to zero. Clearly, $f - h_{\lambda,c}$ has such an extremum if and only if the graph of $h_{\lambda,c}$ supports the graph of f at a point x. Because $f - h_{\lambda,c}$ is continuous, $\{x: f(x) = h_{\lambda,c}(x)\}$ is a nonempty closed set for $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$. Let $E_{f,\lambda}$ be the set of numbers $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda})$ such that $\{x: f(x) = h_{\lambda,c}(x)\}$ is not a perfect set. By Lemma 5, $E_{f,\lambda}$ is dense in $(\alpha_{f,\lambda}, \beta_{f,\lambda})$.

Let $c_0 \in E_{f,\lambda}$ and let x_0 be a point at which a proper maximum of the function $f - h_{\lambda,c_0}$ is equal to zero. For the pair (c_0, x_0) there exists an open rational interval $I \subset [0,1]$ such that $x_0 \in I$, $f(x_0) = h_{\lambda,c_0}(x_0)$ and $f(x) < h_{\lambda,c_0}(x)$ for $x \in I \setminus \{x_0\}$. It is easy to see that the pair (c_0, x_0) is unique. Similarly we prove that if a proper minimum of $f - h_{\lambda,c_0}$ at x_0 is equal to zero, then for (c_0, x_0) there exists an open rational interval $I \subset [0,1]$ such that $x_0 \in I$, $f(x_0) = h_{\lambda,c_0}(x_0)$ and $f(x) >_{\lambda,c_0}(x)$ for $x \in I \setminus \{x_0\}$; moreover, for I the pair (c_0, x_0) is unique. Hence, the set of pairs (c, x), for which $f - h_{\lambda,c}$ has, at x, the proper extremum equal to zero, is countable. This proves that $E_{f,\lambda}$ is countable.

Let $\lambda \in R \setminus \Lambda_f$. For $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$ the graph of $h_{\lambda,c}$ supports the graph of f at most at one point. Hence $\{x: f(x) = h_{\lambda,c}(x)\}$ contains at most one isolated point. If $c \in \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ then, according to Lemma 4, $\{x: f(x) = h_{\lambda,c}(x)\}$ consists of a single point. Moreover, $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda}) \setminus E_{f,\lambda}$; thus $\{x: f(x) = h_{\lambda,c}(x)\}$ is a perfect set. Let $c \in E_{f,\lambda}$ and let x' be an isolated point of $\{x: f(x) = h_{\lambda,c}(x)\}$. From $\lambda \notin \Lambda_f$, it follows that the set $\{x: f(x) = h_{\lambda,c}(x)\}$ contains a unique isolated point. Evidently, $\{x: f(x) = h_{\lambda,c}(x)\} \setminus \{x'\} \neq \varnothing$, and this finishes the proof of 1°.

Let $\lambda \in \Lambda_f$. According to Lemma 7, for $c \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ contains at most two isolated points; by Lemma 8 there exists a unique number $c_{f,\lambda} \in [\alpha_{f,\lambda}, \beta_{f,\lambda}]$ such that $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ contains two isolated points y_1 and y_2. If $c_{f,\lambda} \in \{\alpha_{f,\lambda}, \beta_{f,\lambda}\}$ then, by Lemma 4, $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\}$ consists of two different points. If $c_{f,\lambda} \in E_{f,\lambda}$, then $\{x: f(x) = h_{\lambda,c_{f,\lambda}}(x)\} \setminus \{y_1, y_2\}$ is nonempty and, for every $c \in E_{f,\lambda} \setminus \{c_{f,\lambda}\}$, $\{x: f(x) = h_{\lambda,c}(x)\}$ contains a unique isolated point. If $c \in \{\alpha_{f,\lambda}, \beta_{f,\lambda}\} \setminus \{c_{f,\lambda}\}$ then, according to Lemma 4, $\{x: f(x) = h_{\lambda,c}(x)\}$ consists of a single point. Moreover, for $c \in (\alpha_{f,\lambda}, \beta_{f,\lambda}) \setminus E_{f,\lambda}$ the set $\{x: f(x) = h_{\lambda,c}(x)\}$ is perfect. This proves 2°.

4. The structure of the sets $\{x: f(x) = h(x)\}$ for $f \in C$ and h that is a homeomorphical image of a Lipschitz function. Let φ: $[0,1] \times R \to [0,1] \times R$ be a bijection transform such that for every $a \in [0,1]$ there exists $a' \in [0,1]$ such that

$$\varphi(\{(x, y): x = a, y \in R\}) = \{(x, y): x = a', y \in R\}.$$

The transform φ is defined by $\varphi(x, y) = (g(x), s_x(y))$, where g and s_x are horizontal and vertical sections of φ respectively. Clearly, g: $[0,1] \to [0,1]$ and s_x: $R \to R$ for every $x \in [0,1]$. It is easy to prove the following:

10. LEMMA. *The transform φ is a homeomorphism if and only if*:

1°. *g and s_x are homeomorphisms for every $x \in [0,1]$, and*

2°. *if a sequence $\{x_n\} \subset [0,1]$ converges to x, then $\{s_{x_n}\}$ is continuously convergent to s_x and $\{s_{x_n}^{-1}\}$ is continuously convergent to s_x^{-1}.*

Let $f \in C$ and let

$$W_f = \{(x, y): y = f(x), 0 \leqslant x \leqslant 1\}$$

be the graph of f. The image of W_f in φ is

$$W = \{(g(x), s_x(y)): y = f(x), 0 \leqslant x \leqslant 1\}.$$

W is the graph of the function f^* in C defined by

$$f^*(x) = s_{g^{-1}(x)}(f(g^{-1}(x))).$$

Let $\psi: C \to C$ be the transform such that $\psi(f) = f^*$ for every $f \in C$. It is easy to see that ψ is the bijection transform and $\psi(C) = C$. The following lemma is easy to prove:

11. LEMMA. *The transform ψ is a homeomorphism if and only if $\{s_x\}_{x \in [0,1]}$ and $\{s_x^{-1}\}_{x \in [0,1]}$ are families of equicontinuous functions.*

12. LEMMA. *Let $\psi: C \to C$ be a homeomorphism. Then H is a 2-parameter family of continuous functions if and only if $H^* = \psi(H)$ is a 2-parameter family of continuous functions.*

PROOF. Let H be a 2-parameter family of continuous functions and let $A = (x_1, y_1)$, $B = (x_2, y_2)$ belong to $[0,1] \times R$. The points $A_1 = \varphi^{-1}(A)$ and $B_1 = \varphi^{-1}(B)$ belong to the graph of a unique function h in H. Evidently, A and B belong to the graph of the function $h^* = \psi(h)$. We assume A and B belong to graphs of two different functions h^* and h_1^* in H^*. Because ψ is a homeomorphism, the functions $h_1 = \psi^{-1}(h_1^*)$ and $h = \psi^{-1}(h^*)$ belong to H and $h \neq h_1$. A_1 and B_1 belong to the graphs of h_1 and h. This contradicts the assumption that H is a 2-parameter family. Hence, $h^* = h_1^*$ and H^* is a 2-parameter family. Similarly we prove the sufficiency.

13. LEMMA. *Let $\psi: C \to C$ be a homeomorphism. Then the graph of a function $h \in H$ supports the graph of a function $f \in C$ in an interval $I \subset [0,1]$ if and only if the graph of a function $h^* = \psi(h)$ supports the graph of a function $f^* = \psi(f)$ in an interval $g(I) = I^*$.*

PROOF. We prove the necessity. Let the graph of h support the graph of f in I from above. There exists a point $x_0 \in I$ such that $f(x_0) = h(x_0)$ and $f(x) < h(x)$ for every $x \in I \setminus \{x_0\}$. We have $g(x_0) \in g(I)$ and the point $\varphi(x_0, f(x_0))$ belongs to the graphs of h^* and f^*. Hence, $h^*(g(x_0)) = f^*(g(x_0))$. For every $x \in I \setminus \{x_0\}$ the set $\{y: f(x) < y < h(x)\}$ and $s_x(\{y: f(x) < y < h(x)\})$ are open nonempty intervals. To show that the graph of h^* supports the graph of f^* in $g(I)$, it suffices to prove that for every $x' = g(x) \in g(I) \setminus \{g(x_0)\}$ we have $f^*(x') < h^*(x')$ or $f^*(x') > h^*(x')$. Let x_1 and x_2 be different points in $g(I)$ such that $f^*(x_1) < h^*(x_1)$ and $f^*(x_2) > h^*(x_2)$. There exists a function h_1^* in H^* such that points $(x_1, f^*(x_1))$ and $(x_2, f^*(x_2))$ belong to the graph of h_1^* and there exists a point x_3 between x_1 and x_2 such that $h_1^*(x_3) = h^*(x_3)$. For $h_1 = \psi^{-1}(h_1^*)$ we have $h_1(g^{-1}(x_1)) < h(g^{-1}(x_1))$, $h_1(g^{-1}(x_2)) < h(g^{-1}(x_2))$ and $h_1(g^{-1}(x_3)) = h(g^{-1}(x_3))$. This contradicts the assumption that $h_1, h \in H$.

Similarly we show that if the graph of h supports the graph of f in I from below, then the graph of h^* supports the graph of f^* in $g(I)$.

The proof of the sufficiency follows similarly.

2. THEOREM. *If H is a 2-parameter family for which Theorem 1 holds and ψ is a homeomorphism, then, for a family $H^* = \psi(H)$, Theorem 1 holds.*

PROOF. Let A denote the set of functions determined for H by Theorem 1. The set $A_1 = \psi(A)$ is residual in C. By Lemma 12, H^* is a 2-parameter family of continuous functions. Let B^*, D^* and F^* denote sets of functions determined for H^* by Lemmas 6, 7 and 8, respectively, and let

$$A^* = A_1 \cap B^* \cap (C \setminus D^*) \cap (C \setminus F^*).$$

Clearly, A^* is residual in C. Let $f^* \in A^*$. We denote by $\Lambda^*_{f^*}$ the set of increases of functions h^* in H^* for which the graphs support the graph of f^* in at least two disjoint open subintervals of $[0, 1]$. According to Lemmas 4 and 6, $\lambda^*_{f^*}$ is a countable dense subset in R. For $\lambda^* \in R$ we set

$$\alpha^*_{f^*,\lambda^*} = \inf\{c \in R: \{x: f^*(x) = h^*_{\lambda^*,c}(x)\} \neq \varnothing,\, h^*_{\lambda^*,c} \in H^*\},$$

$$\beta_{f^*,\lambda^*} = \sup\{c \in R: \{x: f^*(x) = h^*_{\lambda^*,c}(x)\} \neq \varnothing,\, h^*_{\lambda^*,c} \in H^*\}.$$

For $\lambda^* \in R$ and $c \in [\alpha^*_{f^*,\lambda^*}, \beta^*_{f^*,\lambda^*}]$ we have

$$\{x: f^*(x) = h^*_{\lambda^*,c}(x)\} = g(\{x: f(x) = h_{\lambda,c}(x)\})$$

when $f = \psi^{-1}(f^*)$ and $h_{\lambda,c} = \psi^{-1}(h_{\psi^*,c^*})$. Because g is a homeomorphism, x is an isolated point of $\{x: f^*(x) = h^*_{\lambda^*,c^*}(x)\}$ if and only if $f - h_{\lambda,c}$ has a proper extremum at $x' = g^{-1}(x)$ equal to zero. $f - h_{\lambda,c}$ has such an extremum if and only if the graph of $h_{\lambda,c}$ supports the graph of f at x'. By Lemma 13 the graph of f^* supports the graph of $h^*_{\lambda^*,c^*}$ at x and $f^* - h^*_{\lambda^*,c^*}$ has a proper extremum at x equal to zero. Clearly, $\{x: f^*(x) = h^*_{\lambda^*,c^*}(x)\}$ is a closed nonempty set for every $c^* \in [\alpha^*_{f^*,\lambda^*}, \beta^*_{f^*,\lambda^*}]$. Let $E^*_{f^*,\lambda^*}$ be the set of numbers $c^* \in (\alpha^*_{f^*,\lambda^*}, \beta^*_{f^*,\lambda^*})$ such that $\{x: f^*(x) = h^*_{\lambda^*,c^*}(x)\}$ is not a perfect set, which occurs if and only if there is a point $x \in \{x: f^*(x) = h^*_{\lambda^*,c^*}(x)\}$ such that $f^* - h^*_{\lambda^*,c^*}$ has a proper extremum at x equal to zero. In the same way as in the proof of Theorem 1 we can show that $E^*_{f^*,\lambda^*}$ is a countable dense set in $(\alpha^*_{f^*,\lambda^*}, \beta^*_{f^*,\lambda^*})$ and, for f^* and $\Lambda^*_{f^*}$, $E^*_{f^*,\lambda^*}$, conditions 1° and 2° of Theorem 1 hold.

From Theorem 2 and Lemmas 10 and 11 we obtain the following corollaries:

2. COROLLARY. *Let H be a 2-parameter family of continuous functions for which Theorem 1 holds and let $t \in C$. Then*

(a) *for the family $H_1 = t + H = \{t + h: h \in H\}$, Theorem 1 holds;*

(b) *if $t(x) \neq 0$ for every $x \in [0, 1]$, then for the family $H_2 = t \cdot H = \{t \cdot h: h \in H\}$, Theorem 1 holds.*

3. COROLLARY. *Let $t \in C$ be a strictly monotone function and let H be a 2-parameter family of continuous functions such that*

$$\underset{\lambda, c \in R}{\forall}\ \underset{L_{\lambda,c} \geqslant 0}{\exists}\ \underset{x_1, x_2 \in [0,1]}{\forall} |h_{\lambda,c}(x_1) - h_{\lambda,c}(x_2)| \leqslant L_{\lambda,c}|t(x_1) - t(x_2)|,$$

and, for every natural number n,

$$M_n = \sup\{L_{\lambda,c}: \lambda, c \in [-n, n]\} < +\infty.$$

Then, for H, Theorem 1 holds.

References

1. A. M. Bruckner and K. M. Garg, *The level structure of a residual set of continuous functions*, Trans. Amer. Math. Soc. **232** (1977), 307–321.

2. ______, *The level structure of typical continuous functions*, Real Anal. Exchange **2** (1976), 35–39.

3. Á. Császár, *Sur la structure des ensembles de niveau des functions réelles á une variable*, Colloq. Math. **4** (1956), 13–29.

4. Z. Wójtowicz, *O typowych postaciach zbiorów* $\{x: f(x) = h(x)\}$ *dla pewnych klas funkcji ciągłych*, Słupskie Prace Matematyczno-Przyrodnicze **3** (1984), 23–45.

Zakład Matematyki, Wyższa Szkoła Pedagogiczna, Słupsk, Polska